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1993 J. Phys. A: Math. Gen. 26 2945

(http://iopscience.iop.org/0305-4470/26/12/030)

J. Phys. A: Math. Gen. 26 (1993) 2945–2953. Printed in the UK

Self-dual Chern–Simons solitons with non-compact groups

D Cangemi

Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Received 4 August 1992, in final form 2 March 1993

Abstract. It is shown how to couple non-relativistic matter with a Chern–Simons gauge field that belongs to a non-compact group. We treat in some detail the semi-simple $SL(2,\mathbb{R})$ and the non-semi-simple Poincaré $ISO(2,1)$ groups. For suitable self-interactions, we are able to exhibit soliton solutions which obey self-dual equations.

1. Introduction

The interaction of non-relativistic planar matter fields with Chern–Simons gauge fields has been extensively studied in recent years [1]. When self-interactions are suitably chosen, and the gauge group is compact, the system admits static solutions fulfilling a set of self-dual equations. The completeness of these static solutions has been discussed in [2]. A natural question addresses the generalization to more general gauge groups. In this paper we present a framework that encompasses models with compact as well as non-compact and non-semi-simple gauge groups. In section 2 we generalize the notion of a Killing form that we need to define the system. In section 3 we show that the reduction of the four-dimensional Yang–Mills self-dual equations leads to static solutions of our problem, provided the matter fields are taken in the adjoint representation. In section 4 we treat as examples the semi-simple $SL(2\mathbb{R})$ group and the non-semi-simple Poincaré group, $ISO(2,1)$. In both cases, special ansätze give explicit solutions to the static problem. Concluding remarks are given in section 5, while an appendix recalls some useful tools in Lie algebra theory.

2. Generalization to non-compact Lie algebras

Let us first recall the form taken by the Lagrangian density in the case of compact Lie algebras (i.e. Lie algebra of a compact Lie group):

$$\mathcal{L} = \tfrac{1}{4}\kappa\epsilon^{\alpha\beta\gamma}\operatorname{tr} A_\alpha\left(\partial_\beta A_\gamma + \tfrac{1}{3}[A_\beta, A_\gamma]\right) + \mathrm{i}\psi^* D_t\psi - \tfrac{1}{2}(\boldsymbol{D}\psi)^*\boldsymbol{D}\psi \\ - \tfrac{1}{2}g\sum_a(\psi^* T_a\psi)(\psi^* T_a\psi). \tag{1}$$

$\{T_a\}$ are the generators of the algebra and $\operatorname{tr} T_a T_b \propto \delta_{ab}$ is its Killing form. As usual, $D_t = \partial_t + A_0$, $\boldsymbol{D} = \boldsymbol{\partial} - \boldsymbol{A}$ are the covariant derivatives, and the matter field ψ is an n-tuplet that transforms according to some finite-dimensional representation of the group. In the generalization of this expression, we preserve the two main properties of $\int \mathrm{d}^3x\mathcal{L}$, namely its reality and its gauge invariance. In order to do so we replace the Killing form and the inner

0305-4470/93/122945+09\$07.50

product in the representing vector space by suitable non-degenerate, Hermitian (for reality) and invariant (for gauge invariance) bilinear forms. Finite-dimensional representations of non-compact groups cannot be unitary; hence we do not expect these forms to be positive definite. Moreover, they do not exist in every representation (see appendix).

Suppose that the adjoint representation possesses such a bilinear form denoted by $\langle T_a, T_b\rangle_{\mathrm{adj}} = \Omega_{ab}$. Suppose also that the matter fields belong to some representation with its own bilinear form $\langle , \rangle$. If Ω^{ab} is the inverse matrix of Ω_{ab}, the natural generalization of equation (1) is

$$\mathcal{L} = \tfrac{1}{4}\kappa\epsilon^{\alpha\beta\gamma}\left\langle A_\alpha, \partial_\beta A_\gamma + \tfrac{1}{3}[A_\beta, A_\gamma]\right\rangle_{\mathrm{adj}} + \mathrm{i}\langle\psi, D_t\psi\rangle - \tfrac{1}{2}\langle \boldsymbol{D}\psi, \boldsymbol{D}\psi\rangle \\ - \tfrac{1}{2}g\sum_{a,b}\langle\psi, T_a\psi\rangle\Omega^{ab}\langle\psi, T_b\psi\rangle. \tag{2}$$

Then most of the discussion in [1] can be followed in this more general case.

The equations of motion read ($\epsilon_{12} = 1$):

$$F^a_{xy} = \partial_x A^a_y - \partial_y A^a_x + [A_x, A_y]^a = -\frac{\mathrm{i}}{\kappa}\Omega^{ab}\langle\psi, T_b\psi\rangle \tag{3a}$$

$$F^a_{i0} = \frac{1}{2\kappa}\epsilon_{ij}\Omega^{ab}\left(\langle\psi, T_b D_j\psi\rangle - \langle D_j\psi, T_b\psi\rangle\right) \tag{3b}$$

$$\mathrm{i}\partial_t\psi = -\tfrac{1}{2}\boldsymbol{D}^2\psi - \mathrm{i}A_0\psi + g\langle\psi, T_a\psi\rangle\Omega^{ab}T_b\psi. \tag{3c}$$

Taking equation (3a) as a definition of $\boldsymbol{A}$, the last equation can also be derived from the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}\int \mathrm{d}^2\boldsymbol{r}\,\left(\langle\boldsymbol{D}\psi, \boldsymbol{D}\psi\rangle + g\langle\psi, T_a\psi\rangle\Omega^{ab}\langle\psi, T_b\psi\rangle\right) \\ &= \frac{1}{2}\int \mathrm{d}^2\boldsymbol{r}\,\left(\langle D_\epsilon\psi, D_\epsilon\psi\rangle + \left(g + \epsilon\frac{1}{\kappa}\right)\langle\psi, T_a\psi\rangle\Omega^{ab}\langle\psi, T_b\psi\rangle\right) \end{aligned} \tag{4}$$

where the last equality involves the definition $D_\epsilon \equiv D_x + \mathrm{i}\epsilon D_y$ ($\epsilon = \pm$) and the discarding of a boundary term.

We can list the other conserved quantities generating symmetries in the system [1]:

$$\begin{aligned} P^i &= \int \mathrm{d}^2\boldsymbol{r}\, T^{0i} && \text{momentum} \\ J &= \int \mathrm{d}^2\boldsymbol{r}\,\epsilon_{ij} r^i T^{0j} && \text{angular momentum} \\ G^i &= tP^i - \int \mathrm{d}^2\boldsymbol{r}\, r^i\langle\psi, \psi\rangle && \text{Galilean boost} \\ D &= tH - \frac{1}{2}\int \mathrm{d}^2\boldsymbol{r}\, r^i T^{0i} && \text{dilation} \\ K &= -t^2 H + 2tD + \frac{1}{2}\int \mathrm{d}^2\boldsymbol{r}\, r^2\langle\psi, \psi\rangle && \text{conformal weight.} \end{aligned} \tag{5}$$

In this system the momentum density T^{0i} corresponds to a current:

$$T^{0i} = -\mathrm{i}\left(\langle\psi, D_i\psi\rangle - \langle D_i\psi, \psi\rangle\right)/2. \tag{6}$$

For static solutions, we deduce from the above that P^i, D and especially H have to vanish. With the special choice $g = -\epsilon/\kappa$, the condition $H = 0$ is realized if ψ fulfils the first-order differential equation

$$D_\epsilon \psi = 0. \tag{7}$$

It is easy to see that the solutions of equations ($3a$) and (7), together with

$$A_0^a = -\epsilon \frac{\mathrm{i}}{2\kappa} \Omega^{ab} \langle \psi, T_b \psi \rangle \tag{8}$$

are time-independent solutions of equation (3). But, unlike the compact case, the converse is not true. Indeed, if $\langle , \rangle$ is non-positive definite (as in the non-compact case) we cannot conclude that equation (7) is the only way to achieve $H = 0$ in equation (4).

3. Reduction of the Yang–Mills self-dual equation

It has already been pointed out that static solutions of a Chern–Simons system are closely related to a reduction of a self-dual equation expressed in four dimensions. This section provides a derivation of this fact for arbitrary Lie algebras. We use either the O(4) or the O(2, 2) invariant metric to raise and lower indices. The self-dual Yang–Mills equation is ($\epsilon^{1234} = 1$)

$$F^{\mu\nu} = \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \qquad F_{\alpha\beta} = \partial_\alpha W_\beta - \partial_\beta W_\alpha + [W_\alpha, W_\beta] \tag{9}$$

W_α being the gauge potentials with value in the Lie algebra [3]. The reduction to two dimensions is achieved by imposing translation invariance with respect to x^3 and x^4 [4]. Take κ positive and write

$$\begin{aligned} x &= x^1 \qquad y = x^2 \qquad \Psi = \sqrt{\tfrac{1}{2}\kappa}(W_3 - \mathrm{i}W_4) \\ A_x &= W_1 \qquad A_y = W_2 \qquad \bar{\Psi} = -\sqrt{\tfrac{1}{2}\kappa}(W_3 + \mathrm{i}W_4). \end{aligned} \tag{10}$$

In these variables and with the definitions $\partial_\pm = \partial_x \pm \mathrm{i}\partial_y$, $A_\pm = A_x \pm \mathrm{i}A_y$ and D_ϵ as in equation (4), equation (9) reduces to

$$\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = (2/\kappa)[\bar{\Psi}, \Psi] \tag{11a}$$

$$D_\epsilon \Psi = 0 \tag{11b}$$

where in the last equation ϵ is correlated with the metric: $\epsilon = +1$ for O(4), and $\epsilon = -1$ for O(2, 2).

Introducing $\mathcal{A}_+ = A_+ + \sqrt{2/\kappa}\Psi$, $\mathcal{A}_- = A_- - \sqrt{2/\kappa}\bar{\Psi}$ we see that if (and only if) $\epsilon = -1$, equations (11) are equivalent to a zero curvature condition

$$\partial_- \mathcal{A}_+ - \partial_+ \mathcal{A}_- + [\mathcal{A}_-, \mathcal{A}_+] = 0. \tag{12}$$

For compact groups, Dunne [2] has found explicitly all the solutions of this last equation. However, it is not clear whether this construction works for non-compact groups. Namely, equation (12) is solved in a matrix representation and the solution is a matrix which does not necessarily belong to the algebra in which we are interested. For example, we are not able to exhibit a solution by this method for the $ISO(2,1)$ case.

We observe that the reduced self-dual equations (11) are the same as equations ($3a$) and (7) provided *we take the matter fields in the adjoint representation*, i.e.

$$\Psi = \sum_a \psi^a T_a \qquad \bar{\Psi} = -\sum_a (\psi^a)^* T_a. \tag{13}$$

Henceforth we shall work with this representation. Moreover, in the compact case, only the choice $\epsilon = -1$ leads to regular solutions. In that case, equation (12) is relevant and one can follow the general discussion in [1] involving chiral currents and give explicit solutions. But in the non-compact case, different signs conspire to ensure the existence of regular solutions only in the opposite case, $\epsilon = +1$, where equation (12) is no longer valid. In our following illustrations we shall only consider this case.

4. *Soliton solutions in the adjoint representation*

We now take two examples of non-compact groups: the semi-simple $SL(2,\mathbb{R})$ and the non-semi-simple $ISO(2,1)$. The matter field is in the adjoint representation and we shall present different ansätze to solve the self-dual equations with $\epsilon = 1$.

4.1. The $SL(2,\mathbb{R})$ case

As discussed in the appendix, the adjoint representation carries a Killing form $\Omega = \mathrm{diag}(1,-1,-1)$. In order to obtain simple differential equations from equation (11)—namely to avoid the commutator on the left-hand side—we try a solution with the gauge field A in a maximal commutative subalgebra; there are two possible choices: $A \propto J_0$ and $A \propto J_2$ (or J_1).

Let us first try the following ansatz:

$$\Psi = u^0 J_0 + u^+ \frac{1}{\sqrt{2}}(J_1 + \mathrm{i}J_2) + u^- \frac{1}{\sqrt{2}}(J_1 - \mathrm{i}J_2) \qquad A_+ = \omega J_0. \tag{14}$$

If u^+ is non-zero, the self-dual equations (with $\epsilon = 1$) become

$$u^0 = 0 \qquad \partial_+(u^+u^-) = 0 \qquad \omega = \mathrm{i}\partial_+ \ln u^+ \tag{15a}$$

$$\nabla^2 \ln |u^+|^2 = -\frac{2}{\kappa}\left(|u^+|^2 - |u^-|^2\right). \tag{15b}$$

We know [1] that regular solutions are found only if $u^- = 0$. The last equation is then the Liouville equation for the norm of u^+. Its phase is fixed (up to a gauge transformation) by requiring regularity for ω. The radially symmetric solutions are

$$u^+ = 2\sqrt{\kappa}N\frac{1}{r}\frac{1}{(r/r_0)^N + (r_0/r)^N}\mathrm{e}^{\mathrm{i}(1-N)\theta} \tag{16a}$$

$$\omega = -4\mathrm{i}N\frac{1}{r}\frac{(r/r_0)^N}{(r/r_0)^N + (r_0/r)^N}\mathrm{e}^{-\mathrm{i}\theta}. \tag{16b}$$

This solution carries an angular momentum $J = -\kappa 2N$ and a conformal weight $K = -\pi\kappa r_0^2 \operatorname{cosec}(\pi/N)$ (note the opposite sign with respect to the compact SU(2) case). With the other choice for ϵ we would have found the opposite sign in the Liouville equation leading to no regular solution.

Consider now the other possibility:

$$\Psi = u^0 J_0 + u^+ \frac{1}{\sqrt{2}}(J_1 + J_0) + u^- \frac{1}{\sqrt{2}}(J_1 - J_0) \qquad A_+ = \omega J_2. \tag{17}$$

The self-dual equations become

$$u^+ u^- = C(x^-) \tag{18a}$$

$$\omega = \partial_+ \ln u^+ \tag{18b}$$

with an arbitrary complex function $C(x^-)$. The combination $\phi = 2\arg u^+ + \arg C$ obeys the sine–Gordon equation in Euclidean space:

$$\nabla^2 \phi = -\frac{2}{\kappa}|C| \sin\phi. \tag{19}$$

To solve it explicitly we take C constant and we find 'multi-kink' solutions, regular everywhere [5]. However, they do not lead to a function ω decreasing at infinity, unless $C = 0$. In that case the solution is

$$\phi = \text{constant} \qquad u^+ = |u^+| e^{i\phi} \qquad \omega = \partial_+ \ln |u^+| \tag{20}$$

which is gauge equivalent to the trivial solution $u^+ = \omega = 0$ and thus gives no new soliton solution.

Equation (16) gives regular radially symmetric solutions. In the compact SU(2) case, the gauge field is chosen in the Cartan subalgebra and this choice, in fact, gives all the regular solutions [2]. It is striking to note that, here, the Cartan subalgebra corresponds to the first case, $A \propto J_0$, the other one leading to trivial solutions. This similarity with the compact SU(2) case is probably related to the fact that $SL(2,\mathbb{R})$ is a real form of $SU(2)^{\mathbb{C}}$. We also expect that all regular radially symmetric solutions are obtained through ansatz (14).

4.2. The $ISO(2,1)$ case

The Poincaré group is an example of a non-compact and non-semi-simple Lie group. The six generators and the bilinear form Ω of the adjoint representation are described in the appendix. For simplicity, we try to take the gauge field in a maximal Abelian subalgebra. Again we have two possibilities. Let us make the first ansatz:

$$\Psi = u^0 J_0 + u^+ \frac{1}{\sqrt{2}}(J_1 + iJ_2) + u^- \frac{1}{\sqrt{2}}(J_1 - iJ_2) + v^0 P_0 + v^+ \frac{1}{\sqrt{2}}(P_1 + iP_2) + v^- \frac{1}{\sqrt{2}}(P_1 - iP_2) \tag{21}$$

$$A_+ = \omega J_0 + e^0 P_0.$$

Always with $\epsilon = 1$, the self-dual equation implies for u^+, u^- and ω the same equations (15) as in the previous example. Regular solutions were obtained only with $u^- = 0$. With this condition the other equations are

$$u^0 = v^0 = 0 \tag{22a}$$

$$\nabla^2 \mathcal{R}\left[\frac{v^+}{u^+}\right] = -\frac{2}{\kappa}|u^+|^2 \mathcal{R}\left[\frac{v^+}{u^+}\right] \tag{22b}$$

$$e^0 = \mathrm{i}\partial_+ \left(\frac{v^+}{u^+}\right) \tag{22c}$$

$$\partial_+(u^+ v^-) = 0. \tag{22d}$$

At first sight it seems that there are not enough constraints, as equation (22b) determines the real part of (v^+/u^+) but e^0 in equation (22c) also depends on its imaginary part. Nevertheless, by a gauge transformation we can always shift e^0 by the total derivative of a regular and real quantity and set the imaginary part of (v^+/u^+) to what we want. We have used the same kind of reasoning to determine the phase of u^+.

We recognize equation (22b) as the deformation of the Liouville equation (15b) (with $u^- = 0$). Namely, if $|u^+|^2 = \ln(1 + |\phi|^2)$ is the general solution involving some analytical function $\phi(x^+)$, we find the solutions of equation (22b) by making an arbitrary deformation $\phi(x^+) \to \phi(x^+)(1 + \epsilon\psi(x^+))$:

$$|u^+|^2 \mathcal{R}\left(\frac{v^+}{u^+}\right) = \kappa \nabla^2 \left(\frac{|\phi|^2}{1 + |\phi|^2}(\psi + \psi^*)\right). \tag{23}$$

In the 'radially symmetric' case—with $\phi(x^+) \propto (x^+)^{-N}$ and $\psi(x^+) \propto (x^+)^M$—equation (23) reads:

$$\mathcal{R}\left(\frac{v^+}{u^+}\right) = \frac{1}{N}\frac{r^M}{(r/r_0)^N + (r_0/r)^N} \times \left[(M-N)\left(\frac{r}{r_0}\right)^N + (M+N)\left(\frac{r_0}{r}\right)^N\right](a_M \cos M\theta + b_M \sin M\theta). \tag{24}$$

The gauge freedom we have allows a convenient choice for its imaginary part:

$$v^+ = 2\sqrt{\kappa} C_{N,M} \frac{r^{M-1}}{[(r/r_0)^N + (r_0/r)^N]^2}\left[(M-N)\left(\frac{r}{r_0}\right)^N + (M+N)\left(\frac{r_0}{r}\right)^N\right] \mathrm{e}^{\mathrm{i}(1-N-M)\theta} \tag{25}$$

where we have used the expression (16a) for u^+. Equation (22c) then gives

$$e^0 = 4\mathrm{i}NC_{N,M}\frac{r^{M-1}}{[(r/r_0)^N + (r_0/r)^N]^2}\mathrm{e}^{\mathrm{i}(1-M)\theta}. \tag{26}$$

In order to avoid singularities at $r = 0$ and $r = \infty$ we have to restrict the integer values of M to $1 - N \leqslant M \leqslant 1 + N$.

Finally, equation (22d) is trivially solved by $v^- = f(x^-)/u^+$. As the 'radially symmetric' choice we take $f(\bar{z}) = \bar{z}^{\alpha}$ and the regular solution is

$$v^- = C_N \left[\left(\frac{r}{r_0}\right)^N + \left(\frac{r_0 0}{r}\right)^N \right] \left[C_1 \left(\frac{r}{r_0}\right)^L e^{-iL\theta} + C_2 \left(\frac{r_0}{r}\right)^L e^{iL\theta} \right] e^{iN\theta} \quad (27)$$

with an integer $L \geqslant N$. Equations (16), (25)–(27) together with $u^0 = v^0 = u^- = 0$ give a soliton solution to our self-dual problem.

In fact, it is possible to consider a more general ansatz with the gauge field in a larger subalgebra than the maximal Abelian one:

$$\begin{aligned} \Psi &= u^+ \frac{1}{\sqrt{2}}(J_1 + iJ_2) + v^0 P_0 + v^+ \frac{1}{\sqrt{2}}(P_1 + iP_2) + v^- \frac{1}{\sqrt{2}}(P_1 - iP_2) \\ A_+ &= \omega J_0 + e^0 P_0 + e^+ \frac{1}{\sqrt{2}}(P_1 + iP_2) + e^- \frac{1}{\sqrt{2}}(P_1 - iP_2). \end{aligned} \quad (28)$$

First of all we remark that, since the commutators of J_0, P_0 with P_1, P_2 only produce P_1, P_2 terms, the gauge choice previously made for ω, e^0 can still be achieved. Moreover, a gauge transformation parallel to P_1, P_2 transforms e^+, e^- like (Λ is a regular complex function):

$$\begin{aligned} e^+ &\to e^+ + \partial_+ \Lambda + i\omega\Lambda \\ e^- &\to e^- + \partial_+ \Lambda^* - i\omega\Lambda^* \end{aligned} \quad (29)$$

while leaving ω, e^0 unchanged. Thus in a suitable gauge we can also take $e^+ = 0$.

For u^+, v^+, v^-, ω, e^0 the equations are similar to the previous ones. The two remaining equations for the two unknown functions v^0, e^- look rather simple:

$$\partial_+(\partial_- v^0 - v^0 \partial_- \ln |u^+|^2) = 0 \qquad e^- = (u^+)^{-1} \partial_+ v^0. \quad (30)$$

The first one is integrated with the help of two arbitrary functions:

$$v^0 = C_1(x^+)|u^+|^2 + \int^{x^-} dy^- C_2(y^-) |u^+|^2(x^+, y^-). \quad (31)$$

As an explicit example we choose $C_2 = 0$, a constant C_1 and the 'radially symmetric' case:

$$\begin{aligned} v^0 &= 4C_1 \kappa N^2 \frac{1}{r^2} \frac{1}{[(r/r_0)^N + (r_0/r)^N]^2} \\ e^- &= 2iC_1 \sqrt{\kappa} N \frac{1}{r^2} \frac{1}{[(r/r_0)^N + (r_0/r)^N]^2} \left[(1+N) \left(\frac{r}{r_0}\right)^N + (1-N) \left(\frac{r_0}{r}\right)^N \right] e^{iN\theta}. \end{aligned} \quad (32)$$

If we choose the gauge field in another direction in the algebra (e.g. $A \propto J_2$) we would find the same trivial solution as in the $SL(2,\mathbb{R})$ example. The set of equations (16), (25)–(27), (32) gives a large class of soliton solutions in the $ISO(2,1)$ case. However, the conserved quantities (5) give nothing interesting on these solutions. Namely the non-trivial ones are given here by

$$\begin{aligned} J &= -\int d^2 r |u^+|^2 \mathcal{R}\left(\frac{v^+}{u^+}\right) \qquad G^i = \int d^2 r r^i |u^+|^2 \mathcal{R}\left(\frac{v^+}{u^+}\right) \\ K &= -\frac{1}{2} \int d^2 r r^2 |u^+|^2 \mathcal{R}\left(\frac{v^+}{u^+}\right). \end{aligned} \quad (33)$$

But due to the angular dependence of $\mathcal{R}(v^+/u^+)$ (cf equation (24)), $J = K = 0$ and G^i is non-vanishing only for the integer $M = 1$, where e^0 is radially symmetric.

5. Conclusion

We have shown how to couple non-relativistic matter to non-compact Chern–Simons theory. This is not always possible since the matter field must be in a representation that carries an invariant bilinear form. In that case, static equations are nicely related to the reduction of the four-dimensional Yang–Mills self-dual equations. Non-trivial and regular solutions are obtained by specific ansätze where the gauge field is restricted to some subalgebra. In the compact case, it is enough to consider the Cartan subalgebra, i.e. the maximal compact and Abelian subalgebra. This still seems to hold with $SL(2, \mathbb{R})$. However, with $ISO(2, 1)$, we have found explicit solutions taking the gauge field in a subalgebra, which was neither Abelian nor compact, cf equation (28).

The presence of these solitons can be useful to understand Euclidean gravity in two dimensions as a reduction of a Chern–Simons system in three dimensions. Although the matter is taken as non-relativistic, this study can also give some insight into the question of coupling matter, in a gauge-invariant form, to $2+1$ gravity seen as a Chern–Simons theory. Moreover, this study provides a good example where a simple physical system with higher symmetries leads to interesting mathematical objects like the sine–Gordon or the Toda equations.

Acknowledgments

I should like to thank R Jackiw for very helpful comments and V Ruuska for many discussions about algebraic questions. This work is supported in part by funds provided by the US Department of Energy (DOE) under contract DE-AC02-76ER03069, and by the Swiss National Science Foundation.

Appendix

In this appendix we discuss the existence of non-degenerate, Hermitian and invariant bilinear forms in a finite-dimensional representation of a Lie algebra. If u, v belong to a representing vector space and if the generators of the Lie algebra act on them by $u \to Tu$, we are looking for a non-degenerate bilinear form $\langle , \rangle$ such that

$$\langle u, v\rangle = \langle v, u\rangle^* \qquad \langle Tu, v\rangle + \langle u, Tv\rangle = 0. \tag{A1}$$

In matrix notation we write $\langle u, v\rangle = (u^m)^* \Omega_{mn} v^n$ with (Ω_{mn}) invertible and

$$\Omega^\dagger = \Omega \qquad T^\dagger = -\Omega T \Omega^{-1}. \tag{A2}$$

If the algebra is semi-simple, the adjoint representation carries such a form: the Killing form. But for non-semi-simple or for other representations this is not always true.

Let us consider the following examples.

(A) A compact, semi-simple Lie algebra like SU(n). All irreducible representations are unitary, thus in all representations $\Omega \propto I$.

(B) A non-compact, semi-simple Lie algebra. Our prototype is $SL(2, \mathbb{R})$:

$$[J_a, J_b] = \epsilon_{ab}^{c} J_c \tag{A3}$$

with $a, b, c = 0, 1, 2$, $\epsilon_{012} = 1$, $\epsilon^{c}{}_{ab} = \eta^{cc'}\epsilon_{abc'}$, $\eta^{ab} = \mathrm{diag}(1, -1, -1)$. In the three-dimensional adjoint representation we have the Killing form $\Omega = \mathrm{diag}(1, -1, -1)$. In the two-dimensional fundamental representation

$$J_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A4}$$

we have $\Omega = 2\mathrm{i}J_0$.

(C) A non-compact, non-semi-simple Lie algebra. Here we consider the Poincaré algebra $ISO(2, 1)$ $(a, b, c = 0, 1, 2)$:

$$[J_a, J_b] = \epsilon^{c}{}_{ab}J_c \qquad [J_a, P_b] = \epsilon^{c}{}_{ab}P_c \qquad [P_a, P_b] = 0. \tag{A5}$$

In the six-dimensional adjoint representation, it turns out that there is still a bilinear form with the good properties (which is not the Killing form):

$$\langle J_a, J_b\rangle_{\mathrm{adj}} = c_1\eta_{ab} \qquad \langle J_a, P_b\rangle_{\mathrm{adj}} = c_2\eta_{ab} \qquad (c_2 \neq 0) \tag{A6}$$

with η_{ab} being the diagonal matrix $\mathrm{diag}(1, -1, -1)$. But this is not true in all representations. For example, in the four-dimensional fundamental one (with $\hat{J}_a$ given by (A4)):

$$J_a = \begin{pmatrix} & & & 0 \\ & \hat{J}_a & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_0 = \begin{pmatrix} & & & 1 \\ & 0 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$P_1 = \begin{pmatrix} & & & 0 \\ & 0 & & 1 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} & & & 0 \\ & 0 & & 0 \\ & & & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A7}$$

we cannot find an invertible matrix Ω with the properties (A2). On the other hand, there is another four-dimensional representation given in terms of 4×4 gamma matrices Γ^A $(A = 0, 1, 2, 3)$ of the four-dimensional Minkowskian space ($\Gamma_A = \eta_{AB}\Gamma^B$, $\Gamma^5 = \mathrm{i}\Gamma^0\Gamma^1\Gamma^2\Gamma^3$):

$$J_a = -\tfrac{1}{4}\epsilon_{abc}\Gamma^b\Gamma^c \qquad P_a = \mathrm{i}\beta\Gamma_a(1 + \Gamma^5) \qquad (\beta \in \mathbb{R}) \tag{A8}$$

which carries the bilinear form $\Omega = \Gamma_0$.

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